

A Candidate for the Abelian Category of Mixed Elliptic Motives

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Abstract

In this work, we suggest a definition for the category of mixed motives generated by the motive $h^1(E)$ for E an elliptic curve without complex multiplication. We then compute the cohomology of this category. Modulo a strengthening of the Beilinson-Soulé conjecture, we show that the cohomology of our category agrees with the expected motivic cohomology groups. Finally for each pure motive $(\mathrm{Sym}^n h^1(E))(-1)$ we construct families of nontrivial motives whose highest associated weight graded piece is $(\mathrm{Sym}^n h^1(E))(-1)$.

Keywords: arithmetic geometry, cohomology, K -theory, motive, elliptic polylogarithm, L -function, representation of GL_n .

1 Introduction

Although a category of motives for smooth projective varieties, called pure motives, defined via algebraic cycles modulo homological equivalence, has been understood since the 1960s, we are only now starting to understand the outlines of a larger category of mixed motives. For instance, Voevodsky [38], Levine [27], and Hanamura have each independently constructed a *derived* category of mixed motives. However, one can still ask for more; a description of the as yet hypothetical abelian heart of such a category (with respect to the appropriate t -structure), at least in the case of rational coefficients. This appears to be difficult. However, some progress has been made in understanding the abelian subcategory of mixed Tate motives (with rational coefficients). For example, Levine [26] (see also Goncharov [19]) showed that the abelian category of mixed Tate motives could be constructed inside the derived category of mixed motives. A different line of attack was

started earlier by Bloch and Kriz ([7], [8]) who, building on ideas of Beilinson and Deligne, explicitly construct a \mathbb{Q} -graded Hopf algebra $\mathcal{H}_{\mathfrak{M}(T)}$, define the category of mixed Tate motives $\mathfrak{M}(T)$ over a field k as the category of finite dimensional \mathbb{Q} -graded co-representations of $\mathcal{H}_{\mathfrak{M}(T)}$, and show that this category satisfies the major properties that such a category should satisfy¹.

Given the relative success of this ground up approach to mixed Tate motives one can ask whether other useful categories of mixed motives can be constructed that eventually will be understood as subcategories of the full abelian category of mixed motives.

Let E be an elliptic curve without complex multiplication defined over a number field k . Let PEM be the category of pure elliptic motives for E , which by definition is the smallest full Tannakian subcategory of the category of pure motives containing $h^1(E)$. It turns out (see for example [35]) that PEM is categorically equivalent to the category of linear representations of $GL_2(\mathbb{Q})$, i.e. a PEM is generated by irreducible representations of the form $\text{Sym}^n h^1(E)(m) := (\text{Sym}^n(h^1(E)) \otimes \mathbb{Q}(m))$. The last step in the standard construction of the category of pure motives involves formally inverting the Tate object $\mathbb{Q}(-1)$. (Note that $\mathbb{Q}(-1)$ has weight 2 while its formal inverse, $\mathbb{Q}(1)$, has weight -2)². Effective pure elliptic motives correspond to motives before the Tate object is inverted i.e. representations $\text{Sym}^n h^1(E)(m)$ with $m \leq 0$. In this manuscript many objects will be either filtered or graded by pure motives i.e. the indexing objects will be linear representations of $GL_2(\mathbb{Q})$. In other words, a filtration by pure elliptic motives is a bifiltration by pairs of integers (n, m) where $n \geq 0$, and a filtration by pure effective motives is a bifiltration by pairs of integers (n, m) with $n \geq 0$ and $m \leq 0$.

In this manuscript we suggest definitions for the category $\widehat{\mathfrak{M}}(E)$ of PEM -filtered mixed elliptic motives and the subcategory $\mathfrak{M}(E)$ of effective mixed elliptic motives, and show that these categories satisfy a number of properties:

1. We explicitly **construct** a PEM -graded Hopf algebra $\chi(\widehat{\mathfrak{M}}(E))$ in the *ind-category of pure elliptic motives* (equivalently, in the ind-category

¹ Some of these properties require a strengthening of the Beilinson-Soulé conjecture, which we know is true for number fields by the work of Borel, Beilinson, and others, but is still conjectural for an arbitrary field.

²The notation is too old to change now. However, in the construction given in this paper it is more natural to index by positively weighted motives rather than negatively weighted motives. Thus in this paper we index by symmetric powers of $h^1(E)$ rather than symmetric powers of $h^1(E)(1)$

of linear GL_2 -representations) from a differential graded algebra $\mathcal{A}_{\widehat{\mathfrak{M}}(E)}$. The differential graded algebra that we define has as an essential component a modified version of the complex used to calculate the higher Chow groups. Let I denote the augmentation ideal of $\chi(\widehat{\mathfrak{M}}(E))$. It turns out that $I/I^2 := \mathcal{M}_{\widehat{\mathfrak{M}}(E)}$ is a Lie co-algebra.

Definition 1.1. *Given an elliptic curve E , the category of mixed elliptic motives is the category of (finite dimensional) comodules over $\chi(\widehat{\mathfrak{M}}(E))$, or equivalently, co-representations of the graded Lie co-algebra $\mathcal{M}_{\widehat{\mathfrak{M}}(E)}$, where the grading is by pure motives.*

2. We then compute the extension groups of this category. Modulo a strengthening of the Beilinson-Soulé conjecture ³, we show (**Proposition 3.1**) that

$$\mathrm{Ext}_{\widehat{\mathfrak{M}}(E)}^1(\mathrm{Sym}^n h^1(E)(-m), \mathbb{Q}) \cong CH^{m+n}(E^n, 2m + n - 1)_{\mathrm{sgn}} \otimes \mathbb{Q},$$

where $A(n)$ denotes $A \otimes \mathbb{Q}(1)^{\otimes n}$ and sgn denotes the sign character eigenspace. (Over a number field these should be the only nonzero extension groups.)

3. We then **construct families of motives** $\mathcal{E}(g_1, \dots, g_n)$ in each $\mathcal{M}_{\mathrm{Sym}^n h^1(E)(-1)}$ (the $\mathrm{Sym}^n h^1(E)(-1)$ - graded piece of $\mathcal{M}_{\widehat{\mathfrak{M}}(E)}$), whose associated weight graded pieces are the pure motives $\mathrm{Sym}^n h^1(E)(-1)$, $\mathrm{Sym}^{n-1} h^1(E)(-1)$, \dots , $h^1(E)(-1)$, $\mathbb{Q}(-1)$, $h^1(E)$, and \mathbb{Q} , and varying over rational functions $g_1, \dots, g_n \in k(E)^*$ whose divisors are pairwise disjointly supported, and pairwise disjoint from the identity element. (Note that this necessitates using at least $2n$ distinct points of $E(k)$.) In general, $\mathcal{E}(g_1, \dots, g_n)$ does not define an extension. A mixed motive is a filtered object, hence it need not be a pure motive or even an extension of pure motives (this is a point that tends to be obscured when one just looks at motivic cohomology). However, linear combinations of these motives define (nontorsion) elements of $\mathrm{Ext}_{\widehat{\mathfrak{M}}(E)}^1(\mathrm{Sym}^n h^1(E)(-1), \mathbb{Q})$.

In [28] we construct the realization functor of $\widehat{\mathfrak{M}}(E)$ to the category of mixed Hodge structures and explicitly calculate the images of the $\mathcal{E}(g_1, \dots, g_n)$'s.

³i.e. We assume $\mathcal{A}_{\widehat{\mathfrak{M}}(E)}$ is a $K(\pi, 1)$ in the sense of Sullivan,

One reason to study an abelian category of motives (rather than relying on a derived category of motives) is that such a category is a finer tool for probing the structure of varieties, and its study elicits structures not detected by the derived category.

This paper was written concurrently with [5]. Both this paper and [5] grew out of work done for my thesis under the direction of S Bloch (as announced in [5]).

In [18] (p. 25) Goncharov cites [5] as a reference for a description of the elliptic motivic Lie co-algebra similar to the ones outlined in [5] and in section 2 of this paper.

The plan of the sections is as follows. In section 2 we define the categories $\mathfrak{M}(E)$ and $\widehat{\mathfrak{M}}(E)$. In section 3 we show that, if $\widehat{\mathfrak{M}}(E)$ is a $K(\pi, 1)$ in the sense of Sullivan, that the Ext groups of our category agree with the expected motivic cohomology groups. In Section 3 we define families of motives with weight graded pieces $\mathrm{Sym}^n h^1(E)(-1)$, $\mathrm{Sym}^{n-1} h^1(E)(-1), \dots, h^1(E)(-1), \mathbb{Q}(-1), h^1(E)$, and \mathbb{Q} .

2 The Motivic DGA

In this chapter we will define the categories $\widehat{\mathfrak{M}}(E)$ of mixed elliptic motives and the full subcategory $\mathfrak{M}(E)$ of effective mixed elliptic motives by explicitly constructing the motivic Hopf algebras $\chi(\widehat{\mathfrak{M}}(E))$ and $\chi(\mathfrak{M}(E)) = \chi$. For expository reasons it will be easier to start with the subcategory of effective mixed elliptic motives and then pass to the larger category of all mixed elliptic motives.

2.1 Some Initial Notation

We will assume that the reader is familiar with the concepts of a minimal model, 1-minimal model, and the bar construction for a commutative differential graded algebra (DGA) A . The concept of a generalized minimal model is due originally to Quillen. In the form used in this paper (extensions by free 1 dimensional models) it is due originally to Sullivan [33]. A good reference for the applications of minimal models we have in mind is the treatment in [22], Part IV. The bar construction is due originally to Eilenberg and Mac Lane. Good references for the use of the bar construction in this paper are [9] and [8] section 2.

Fix once and for all an elliptic curve E without complex multiplication over a number field k :

$$\pi : E \rightarrow \operatorname{Spec}(k)$$

Cycles with \mathbb{Q} -coefficients are to be understood for the remainder of the paper unless otherwise stated.

Let $h : \operatorname{Var}_{\mathbb{Q}} \rightarrow \operatorname{PM}$ be the functor that sends (smooth projective) varieties to the category of pure motives over k . For every n we have a motive $h(E^n) \in \operatorname{PEM} \subset \operatorname{PM}$, where PEM denotes the category of pure elliptic motives.

Note that $\mathbb{Q}(-1)$ is the direct summand $\wedge^2 h^1(E)$ of $h^1(E) \otimes h^1(E)$.

The tensor product of an object of an additive k -linear category \mathcal{C} with a (possibly infinite dimensional) vector space is defined (representable-functorially) as follows: let $A \in \operatorname{Ob}\mathcal{C}$, V a k -vector space. Then for all $B \in \operatorname{Ob}\mathcal{C}$,

$$\operatorname{Hom}(A \otimes V, B) := \operatorname{Hom}(V, \operatorname{Hom}(A, B))$$

Since we want to talk about tensor products of pure motives with infinite dimensional vector spaces, we are really working in the ind-category of pure motives.

Since E is regular, we can (for the purposes of this paper) define motivic cohomology of $X = E \times E \times E \dots \times E$ in terms of the cubical higher Chow groups:

$$H_{\mathcal{M}}^i(X, \mathbb{Z}(j)) := CH^j(X, 2j - i).$$

In this paper we will restrict to the study of rational motivic cohomology.

Let $\tilde{\mathcal{Z}}^a(E^b, \cdot)$ denote the terms in the cubical higher Chow group complex $\otimes \mathbb{Q}$. (Higher Chow groups are defined for example in [7]. For the definition of cubical higher Chow groups see for example Totaro [37])

We will need to impose relations on the cubical higher Chow group complex in order to give this complex the structure of a suitable commutative-graded differential graded algebra.

We define an action of $G_c := (\mathbb{Z}/2\mathbb{Z})^c \rtimes \Sigma_c$ on $\tilde{\mathcal{Z}}^a(E^b, c)$ where Σ_c acts on $(\mathbb{P}^1 - \{1\})^c$ by permutation and $(\mathbb{Z}/2\mathbb{Z})^c$ acts on $(\mathbb{P}^1 - \{1\})^c$ with the j -th coordinate vector acting by $x \mapsto x^{-1}$ on the j -th factor. Let $\operatorname{Alt}_{G_c} = \sum_{\sigma \in G_c} (-1)^{\operatorname{sgn} \sigma} \sigma$ denote the alternating projection with respect to this grading. Then we define

$$\mathcal{Z}^a(E^b, c) := \operatorname{Alt}_{G_c}(\tilde{\mathcal{Z}}^a(E^b, c)).$$

It follows (see for example [6]) that $\sum \mathcal{Z}^a(E^b, c)$ defines a graded-commutative DGA.

2.2 Effective mixed elliptic motives: The DGA $\mathcal{A}_{\mathfrak{M}(E)}$ (and the Motivic Hopf Algebra χ)

We define a left action of

$$G_b := (\mathbb{Z}/2\mathbb{Z})^b \rtimes \Sigma_b$$

on

$$\mathcal{Z}_c^b := \mathcal{Z}^b(E^b, c)$$

as follows: Σ_b acts by permuting the copies of E in E^b . The generators of $(\mathbb{Z}/2\mathbb{Z})^b$ act by $x \mapsto -x$ (in the group law on E) on the appropriate copy of E . We will think of this action as a right action as follows: Let $\sigma \in \mathbb{Q}[\Sigma_b]$ be a generator. Let $|\sigma|$ denote the number of transpositions in a minimal decomposition of σ . Then

$$\mathcal{Z}_c^b \cdot \sigma := ((-1)^{\text{signature}(\sigma)}(\sigma)^{-1})(\mathcal{Z}_c^b) = ((-1)^{(|\sigma|+1)}(\sigma)^{-1})(\mathcal{Z}_c^b).$$

We note for future reference that if $p \in \mathbb{Q}[\Sigma_b]$ is a projector then

$$\mathcal{Z}_c^b \cdot p := p^t(\mathcal{Z}_c^b)$$

where p^t denotes the transpose of the projector p .

We also have a left action by permutations of Σ_b on

$$h^1(E)^{\otimes b} =: h^b$$

Definition 2.1. $\mathcal{A}_{\mathfrak{M}(E)}$ ($= \mathcal{A}$) $:= \sum_i \mathcal{A}_{\mathfrak{M}(E)}^i := \sum_i \sum_b \mathcal{A}_{\mathfrak{M}(E)}^i(b)$, where

$$\begin{aligned} \mathcal{A}_{\mathfrak{M}(E)}^i(b) &:= \text{Alt}_{(\mathbb{Z}/2\mathbb{Z})^b}(\mathcal{Z}^b(E^b, b-i)) \otimes_{\mathbb{Q}[\Sigma_b]} h^1(E)^{\otimes b} \\ &=: \mathcal{Z}_{b-i}^b \boxtimes h^b \end{aligned}$$

Note that $\mathcal{A}_{\mathfrak{M}(E)}^i(b)$ is an (infinite-dimensional) GL_2 -representation and that $\mathcal{A}_{\mathfrak{M}(E)}$ lives in the ind-category of (infinite-dimensional) GL_2 -representations.

We now define an algebra structure on $\mathcal{A}_{\mathfrak{M}(E)}$. Notice that

$$(\mathcal{Z}_{b-i}^b \otimes_{\mathbb{Q}[\Sigma_b]} h^b) \otimes (\mathcal{Z}_{b'-i'}^{b'} \otimes_{\mathbb{Q}[\Sigma_{b'}]} h^{b'}) = (\mathcal{Z}_{b-i}^b \otimes \mathcal{Z}_{b'-i'}^{b'}) \otimes_{\mathbb{Q}[\Sigma_b \times \Sigma_{b'}]} (h^b \otimes h^{b'}) \quad (2.1)$$

since

$$\mathbb{Q}[\Sigma_b] \otimes \mathbb{Q}[\Sigma_{b'}] = \mathbb{Q}[\Sigma_b \times \Sigma_{b'}].$$

We have external product maps on cycles

$$\mathcal{Z}_c^b \otimes_{\mathbb{Q}} \mathcal{Z}_{c'}^{b'} \rightarrow \mathcal{Z}_{c+c'}^{b+b'}; \quad \mathcal{Z}_{c'}^{b'} \otimes_{\mathbb{Q}} \mathcal{Z}_c^b \rightarrow \mathcal{Z}_{c+c'}^{b+b'}$$

and an external product on GL_2 -representations

$$h^b \otimes h^{b'} \rightarrow h^{b+b'}$$

which are compatible with the map $\Sigma_b \times \Sigma_{b'} \rightarrow \Sigma_{b+b'}$ (and hence the map $\mathbb{Q}[\Sigma_b \times \Sigma_{b'}] \rightarrow \mathbb{Q}[\Sigma_{b+b'}]$), i.e.

$$(C_1 g_1) \otimes (C_2 g_2) = (C_1 \cdot C_2)(g_1 \times g_2); \quad (g_1 h_1) \otimes (g_2 h_2) = (g_1 \times g_2)(h_1 \cdot h_2).$$

This induces an algebra structure on $\mathcal{A}_{\mathfrak{M}(E)}$ under the product map

$$\begin{aligned} \mathcal{A}_{\mathfrak{M}(E)}^i(b) \otimes_{\mathbb{Q}} \mathcal{A}_{\mathfrak{M}(E)}^{i'}(b') &\xrightarrow{\delta} \mathcal{A}_{\mathfrak{M}(E)}^{i+i'}(b+b'); \\ \text{where } (C_1 \boxtimes h_1) \otimes_{\mathbb{Q}} (C_2 \boxtimes h_2) &\mapsto (C_1 \cdot C_2) \otimes_{\mathbb{Q}} (h_1 \cdot h_2). \end{aligned} \quad (2.2)$$

Note that $\mathcal{A}_{\mathfrak{M}(E)}$ is not connected with respect to the grading by i . However, (here we follow [8]) note that the *Adams grading*

$$\mathcal{A}_{\mathfrak{M}(E)} = \mathcal{A}_{\mathfrak{M}(E)}(0) \oplus \mathcal{A}_{\mathfrak{M}(E)}(1) \oplus \cdots$$

satisfies the following properties:

1. $\mathcal{A}_{\mathfrak{M}(E)}(0) \cong k$, $\mathcal{A}_{\mathfrak{M}(E)}(i) = 0$ for $i < 0$ (in particular, \mathcal{A} is connected with respect to the Adams grading),
2. The differential ∂ has Adams degree zero i.e., each $\mathcal{A}_{\mathfrak{M}(E)}(i)$ is a sub-complex of $\mathcal{A}_{\mathfrak{M}(E)}$,
3. The Adams grading is compatible with the algebraic structure i.e., $\mathcal{A}_{\mathfrak{M}(E)}(i) \otimes \mathcal{A}_{\mathfrak{M}(E)}(j) \rightarrow \mathcal{A}_{\mathfrak{M}(E)}(i+j)$.

Since the Adams grading (codimension of cycle) gives \mathcal{A} a connected graded structure, we can calculate the 1-minimal model $(\wedge \mathcal{M}_{\mathcal{A}}[-1])$ of \mathcal{A} (in the sense of Sullivan - see for example [33]). Let $B(\mathcal{A})$ denote the bar construction of \mathcal{A} .

Definition 2.2. $\chi = H^0(B(\mathcal{A}_{\mathfrak{M}(E)}^\bullet))$. The associated Lie coalgebra \mathcal{M} is I/I^2 where I denotes the augmentation ideal.

The category $\mathfrak{M}(E)$ of effective mixed elliptic motives is defined to be the category of co-representations of the Lie co-algebra \mathcal{M} , or (equivalently) the category of comodules over χ .

This definition makes the most sense philosophically if \mathcal{A} is a $K(\pi, 1)$. We will say a bit more about the philosophy at the end of section 3.

We actually have a finer graded structure on our algebra than the one given by the Adams grading. Let $\mathcal{Z}^b := \sum_i \mathcal{Z}_{b-i}^b$. Let $p \in \mathbb{Q}[\Sigma_b]$ be a projector (note that projectors are idempotents ($p^2 = p$)).

Then

$$\mathcal{A}_{\mathfrak{M}(E)}(b) \supset \mathcal{Z}^b \boxtimes p \cdot h^b = \mathcal{Z}^b \boxtimes p^2 \cdot h^b = \mathcal{Z}^b \cdot p \boxtimes p \cdot h^b$$

Note that this computation makes sense in the ind-category of pure elliptic motives. (\mathcal{Z}_{b-i}^b is infinite dimensional, but is a direct sum of finite dimensional GL_2 -representations.)

Thus if p_1 and p_2 are two projectors, the product map (2.2) on $\mathcal{A}_{\mathfrak{M}(E)}$ induces a product map (\star) :

$$(\mathcal{Z}^b \cdot p_1 \boxtimes p_1 \cdot h^b) \otimes (\mathcal{Z}^{b'} \cdot p_2 \boxtimes p_2 \cdot h^{b'}) \xrightarrow{\delta} (\mathcal{Z}^{b+b'} \cdot (p_1 \otimes p_2) \boxtimes (p_1 \otimes p_2) \cdot h^{b+b'})$$

As noted at the beginning of this section, the right action of a projector p on the cycle side corresponds to a left action by p^t .

For example, let $\text{Sym}^b(h^1(E))$ denotes the trivial module, Sym_{Σ_b} the projection associated to the trivial projection $\sum_{\sigma \in \Sigma_b} \sigma$, and Alt_{Σ_b} denotes the projection associated to the alternating projection $\sum_{\sigma \in \Sigma_b} (-1)^{|\sigma|+1} \sigma$. Then

$$\begin{aligned} \mathcal{Z}^b \boxtimes \text{Alt}_{\Sigma_b} \cdot h^b &= \mathcal{Z}^b \cdot \text{Sym}_{\Sigma_b} \boxtimes \text{Sym}^b(h^1(E)) \\ &= \text{Alt}_{\Sigma_b}(\mathcal{Z}^b) \boxtimes \text{Sym}^b(h^1(E)) \\ &= \text{Sym}_{\Sigma_b}^t(\mathcal{Z}^b) \boxtimes \text{Sym}^b(h^1(E)) \end{aligned}$$

since $(p^t)^t = p$.

We will feel free at times to suppress writing projection on the cycle side when we write (cycle) \boxtimes (irreducible representation).

We are now in a position to understand the piece of χ graded by the pure motive $p \cdot h^b$.

The proof of the following lemma is straightforward:

Lemma 2.3. *The differential ∂ and the product map δ on cycles commute with the projection induced via an arbitrary Young symmetrizer.*

In particular, the projection induced via an arbitrary Young symmetrizer commutes with the total differential on the bar complex.

It follows that our algebra is graded by irreducible representations of GL_2 . Indeed, we have a direct sum decomposition

$$\mathcal{A}_{\mathfrak{M}(E)}^i(b)(= \mathcal{Z}_{b-i}^b \boxtimes h^b) = \sum_{p_j} \mathcal{Z}_{b-i}^b \boxtimes p_j \cdot h^b \quad (2.3)$$

where the sum runs through a set of projectors p_j such that $\sum_j p_j = \text{Id}$, the identity projector.

Note that every irreducible representation \mathcal{U} of h^b is of the form $p \cdot h^b$ for some projector p .

Note that if p_1 and p_2 are two projectors, $p_1 \neq p_2$, but $p_1 \cdot h^b \cong p_2 \cdot h^b$, we still have $\mathcal{Z}_{b-i}^b \boxtimes p_1 \cdot h^b \neq \mathcal{Z}_{b-i}^b \boxtimes p_2 \cdot h^b$ as the cycles will depend on the choice of projector.

We now remark that the product structure preserves the decomposition of our algebra into pieces labeled by projectors.

Let $q \in \mathbb{Q}[\Sigma_{b+b'}]$ be an projector. It follows that we can define the $q \cdot h^{\otimes b+b'}$ -graded piece of χ :

Definition 2.4. $\chi(q \cdot h^{\otimes b+b'}) := \text{Hom}_{GL(h^1(E))}(q \cdot h^{\otimes b+b'}, \chi)$

componentwise in terms of direct summands of $(\mathcal{A}_{\mathfrak{M}(E)}^{\otimes i})$. We have

$$q \cdot (h(E^b) \otimes h(E^{b'})) = q \cdot ((h^1(E))^{\otimes b} \otimes (h^1(E))^{\otimes b'}) = q \cdot ((h^1(E))^{\otimes b+b'}) = q \cdot (h(E^{b+b'}))$$

which induces the equality

$$(\mathcal{Z}_{b-i}^b \otimes \mathcal{Z}_{b'-i'}^{b'}) \otimes_{\mathbb{Q}[\Sigma_b \times \Sigma_{b'}]} q \cdot (h^b \otimes h^{b'}) = (\mathcal{Z}_{b-i}^b \otimes \mathcal{Z}_{b'-i'}^{b'}) \otimes_{\mathbb{Q}[\Sigma_b \times \Sigma_{b'}]} q \cdot (h^{b+b'})$$

Notice that $q \in \mathbb{Q}[\Sigma_{b+b'}]$ does not act on $(\mathcal{Z}_{b-i}^b \otimes \mathcal{Z}_{b'-i'}^{b'})$ unless $q \in \mathbb{Q}[\Sigma_b \times \Sigma_{b'}]$.

When we combine the above equality with the algebra structure on $\mathcal{A}_{\mathfrak{M}(E)}^\bullet$ the product map restricts as follows:

Let $\mathcal{V} = p_1 \cdot h^b$ and $\mathcal{W} = p_2 \cdot h^{b'}$ denote pure motives, and suppose $\mathcal{U} \subset \mathcal{V} \otimes \mathcal{W}$ is an irreducible summand. Given $X \boxtimes \mathcal{V} \subset \mathcal{Z}_{b-i}^b \boxtimes h^b$ and $Y \boxtimes \mathcal{W} \subset \mathcal{Z}_{b'-i'}^{b'} \boxtimes h^{b'}$, we have

$$(X \boxtimes \mathcal{V}) \otimes (Y \boxtimes \mathcal{W}) \mapsto (X \cdot Y) \boxtimes (\mathcal{V} \otimes \mathcal{W});$$

under the product map (\star) .

If $\mathcal{U} = q \cdot (\mathcal{V} \boxtimes \mathcal{W})$, the algebra map composed with the projection defined by q induces a product

$$(X \boxtimes \mathcal{W}) \otimes (Y \boxtimes \mathcal{V}) \mapsto (X \cdot Y) \boxtimes \mathcal{U};$$

Let

$$((\mathcal{Z}_{b-i}^b \boxtimes \mathcal{V}) \otimes (\mathcal{Z}_{b'-i'}^{b'} \boxtimes \mathcal{W}))_{\mathcal{U}} = (\mathcal{Z}_{b-i}^b \otimes \mathcal{Z}_{b'-i'}^{b'}) (p_1 \times p_2) \otimes_{\mathbb{Q}[\Sigma_b \times \Sigma_{b'}]} q \cdot (p_1 \times p_2) \cdot (h^{b+b'})$$

Then the appropriate piece of the bar complex looks as follows:

$$\begin{array}{ccc} \ddots \uparrow \partial & & \uparrow \partial \\ \xrightarrow{\delta} ((\mathcal{Z}_{b-i}^b \boxtimes \mathcal{V}) \otimes (\mathcal{Z}_{b'-i'}^{b'} \boxtimes \mathcal{W}))_{\mathcal{U}} & \xrightarrow{\delta} & \mathcal{Z}_{b+b'-i-i'}^{b+b'} \boxtimes \mathcal{U} \\ \uparrow \partial & \ddots & \uparrow \partial \\ & \xrightarrow{\delta} & \mathcal{Z}_{b+b'-i-i'+1}^{b+b'} \boxtimes \mathcal{U} \end{array} \quad (2.4)$$

Here the diagonal dots indicate the H^0 diagonal of $B(\mathcal{A}_{\mathfrak{M}(E)}^\bullet)$ (although if χ turns out not to be a $K(\pi, 1)$ then for the purposes of this discussion the dots could represent any H^i diagonal).

In particular the above discussion makes sense if $q \cdot (h^{b+b'})$ is equal to an irreducible pure motive $\mathrm{Sym}^n h^1(E)(-m)$.

Furthermore, when we pass to the Lie co-algebra \mathcal{M} , it therefore makes sense to talk about the $\mathrm{Sym}^n h^1(E)(-m)$ -graded pieces of \mathcal{M} .

Let $I \subset H^0(B(\mathcal{A}_{\mathfrak{M}(E)}))$ denote the augmentation ideal.

Definition 2.5. $\mathcal{M}_{\mathrm{Sym}^n h^1(E)(-m)} := I(\mathrm{Sym}^n h^1(E)(-m))/I^2(\mathrm{Sym}^n h^1(E)(-m))$

Remark 2.6. *In the last chapter of this paper (where we compute explicit motivic elements) there will be no need to pass to the full category of mixed elliptic motives. In that section of the paper we will therefore work in the category $\mathfrak{M}(E)$ of effective mixed elliptic motives.*

2.3 Mixed elliptic motives:

**The Definition of the Motivic DGA $\mathcal{A}_{\widehat{\mathfrak{M}(E)}}$
and of the Motivic Hopf Algebra $\chi(\widehat{\mathfrak{M}(E)})$**

Consider the following algebra (all conventions as in the previous section):

$$\begin{aligned}
\mathcal{A}_{\mathfrak{M}(E)^{aug}}^i(a, b) &:= \text{Alt}_{(\mathbb{Z}/2\mathbb{Z})^b}(\mathcal{Z}^{b-a}(E^b, b-2a-i)) \otimes_{\mathbb{Q}[\Sigma_b]} (h^1(E)^{\otimes b})(a) \\
&=: \mathcal{Z}_{b-2a-i}^{b,a} \boxtimes h_a^b
\end{aligned}$$

Given a positive integer b , we say that a pure elliptic motive \mathcal{V} is *effective of weight b* if it is a direct summand of $h^1(E)^{\otimes b}$. Notice that any pure elliptic motive \mathcal{W} can be written $\mathcal{W} = \mathcal{V}(a)$ as some positive twist $\mathbb{Q}(a)$ of an effective motive \mathcal{V} of weight b for some b . Clearly $(\mathcal{W}(-c))(c) = (\mathcal{V}(-c))(a+c)$, where $(\mathcal{V}(-c))$ is effective of weight $b+2c$ and $c > 0$, is another way to write the same motive \mathcal{W} , and just as clearly these are the only ways to decompose \mathcal{W} into a positive twist of an effective motive.

The above paragraph becomes substantive when we adjoin cycle groups to each pure motive. Namely, $\mathcal{Z}_{n-i}^{n,0} \boxtimes \mathcal{V}$ may be different from $\mathcal{Z}_{n-i}^{n+2a,a} \boxtimes (\mathcal{V}(-a))(a)$, which a priori suggests a well-definedness problem. Thus we modify the construction of $\mathcal{A}_{\mathfrak{M}(E)^{aug}}(a, b)$ by taking a direct limit to remove this ambiguity.

As in the last section, the algebra $\mathcal{A}_{\mathfrak{M}(E)^{aug}}(a, b) (= \sum_i \sum_b \mathcal{A}_{\mathfrak{M}(E)^{aug}}^i(a, b))$ is graded by irreducible representations of GL_2 . Indeed, we have a direct sum decomposition

$$\mathcal{Z}_{b-2a-i}^{b,a} \boxtimes h_a^b = \sum_{p_j} \mathcal{Z}_{b-2a-i}^{b,a} \boxtimes (p_j(h^b))(a) \quad (2.5)$$

where the sum runs through a set of projectors p_j such that $\sum_j p_j = \text{Id}$, the identity projector.

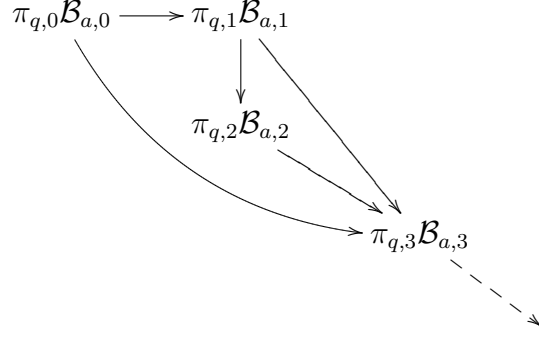
Note that $\sum_j p_j(h^1(E)^{\otimes b})(a) = h^1(E)^{\otimes b}(a)$ only if $a = 0$.

Let $\mathcal{V} = q \cdot h^1(E)^{\otimes b}$ be an effective irreducible representation of GL_2 (with associated projector q), let $\mathcal{B}_{a,c} = \mathcal{Z}_{b-2a-i}^{b+2c,a+c} \boxtimes h_{a+c}^{b+2c}$ and $\pi_{q,c}\mathcal{B}_{a,c} = \mathcal{Z}_{b-2a-i}^{b+2c,a+c} \boxtimes \mathcal{V}(a)$ be the projection. Note that we have inclusion maps $\pi_{q,c}\mathcal{B}_{a,c} \hookrightarrow \mathcal{B}_{a,c}$ and projection maps $\mathcal{B}_{a,c} \twoheadrightarrow \pi_{q,c}\mathcal{B}_{a,c}$ for all c .

We have diagrams

$$\begin{array}{ccccccc}
\pi_{q,0}\mathcal{B}_{a,0} & \hookrightarrow & \mathcal{B}_{a,1} & \hookrightarrow & \mathcal{B}_{a,2} & \hookrightarrow & \cdots \\
& & \uparrow \downarrow & & \uparrow \downarrow & & \\
& & \pi_{q,1}\mathcal{B}_{a,1} & & \pi_{q,2}\mathcal{B}_{a,2} & &
\end{array}$$

which induce



In particular notice that for large enough i , $\pi_{q,i}\mathcal{B}_{a,i} \xrightarrow{\sim} \pi_{q,j}\mathcal{B}_{a,j}$ is a quasi-isomorphism for all $i < j$.

Definition 2.7. *We have*

$$\mathcal{A}_{\widehat{\mathfrak{M}}(E)}^\bullet := \sum_i \lim_{\rightarrow} \left(\sum_b \sum_{0 \leq a < b} \sum_{p_j} \mathcal{A}_{\widehat{\mathfrak{M}}(E)}^i(a, p_j) \right),$$

where

$$\mathcal{A}_{\widehat{\mathfrak{M}}(E)}^i(a, q) := \lim_{\vec{c}} \pi_{q,c} \mathcal{B}_{a,c}.$$

Here the limits are filtered colimits taken over all diagrams of the form listed above, and the sum over p_j runs through a set of projectors p_j such that $\sum_j p_j = \text{Id}$, the identity projector.

Equivalently,

$$\mathcal{A}_{\widehat{\mathfrak{M}}(E)}^\bullet := \sum_i \sum_q \mathcal{A}_{\widehat{\mathfrak{M}}(E)}^i(a, q)$$

where the sum over q runs over all irreducible objects in the category of linear representations of Gl_2 .

Note that since the limits being taken are filtered they will be exact, and so commute with arbitrary sums and cohomology.

Also note that the product structure discussed in the last section extends in a natural way to a product structure on $\mathcal{A}_{\widehat{\mathfrak{M}}(E)}^\bullet$.

Definition 2.8. $\chi = H^0(B(\mathcal{A}_{\widehat{\mathfrak{M}}(E)}^\bullet))$. The associated Lie coalgebra $\widehat{\mathcal{M}}$ is $\widehat{I}/\widehat{I}^2$ where \widehat{I} denotes the augmentation ideal.

The category $\widehat{\mathfrak{M}}(E)$ of mixed elliptic motives is defined to be the category of co-representations of the Lie co-algebra $\widehat{\mathcal{M}}$, or (equivalently) the category of comodules over $\widehat{\chi}$.

Remark 2.9. It follows from Proposition 3.1 from the next chapter that

$$CH^n(E^{2n}, 0) \boxtimes \mathbb{Q} \cong \mathbb{Q}$$

and hence

$$\lim_{\rightarrow} CH^n(E^{2n}, 0) \boxtimes \mathbb{Q} \cong \mathbb{Q}.$$

In other words, the augmentation map still maps to the coefficient ring \mathbb{Q} .

The results from [22] imply the following proposition:

Proposition 2.10. The category $\widehat{\mathfrak{M}}(E)$ is equivalent to the heart $\mathfrak{H}_{\widehat{\mathcal{A}}_{\widehat{\mathfrak{M}}(E)}^\bullet}$ of the derived category $\mathfrak{D}_{\mathcal{A}_{\widehat{\mathfrak{M}}(E)}^\bullet}$ with respect to the t -structure defined in Theorem 1.1 (p. 59) from [22].

It would be very interesting to relate $\mathfrak{D}_{\mathcal{A}_{\widehat{\mathfrak{M}}(E)}^\bullet}$ to the categories defined by Voevodsky, Levine, and Hanamura.

3 A General Cycle Group Computation

In this section we will compute the cohomology of our DGA $\mathcal{A}_{\widehat{\mathfrak{M}}(E)}$. We then relate these cohomology groups to the Ext-groups of our category $\widehat{\mathfrak{M}}(E)$ assuming that $\mathcal{A}_{\widehat{\mathfrak{M}}(E)}$ is a $K(\pi, 1)$.

3.1 Motivic Complexes

In order to compute the Ext-groups of our category, we consider the associated category of representations of the associated Lie co-algebra. A basic result for Lie coalgebras (see for example [39] pp. 224-5) is that the Ext-groups of the category of co-representations of a graded Lie coalgebra, such as \mathcal{M} , can be computed by looking at the complex

$$\mathcal{M} \rightarrow \bigwedge^2 \mathcal{M} \rightarrow \bigwedge^3 \mathcal{M} \dots$$

where the maps are given by the differential on the Lie-coalgebra.

As discussed in the previous section (see (2.5)), our Lie coalgebra is labelled by irreducible linear representations \mathcal{U} of GL_2 . Thus we have a direct sum decomposition

$$\mathcal{M} \cong \oplus (\mathcal{M}_{\mathcal{U}} \otimes \mathcal{U}).$$

Hence $\bigwedge^2 \mathcal{M}$ breaks up into a direct sum decomposition

$$\begin{aligned} \bigwedge^2 \mathcal{M} &\cong \bigwedge^2 (\oplus (\mathcal{M}_{\mathcal{U}} \otimes \mathcal{U})) \cong \\ &\sum (\mathcal{M}_{\mathcal{V}} \bigwedge \mathcal{M}_{\mathcal{W}}) \otimes (\text{Sym}(\mathcal{V} \otimes \mathcal{W})) \bigoplus (\text{Sym}(\mathcal{M}_{\mathcal{V}} \otimes \mathcal{M}_{\mathcal{W}})) \otimes (\mathcal{V} \wedge \mathcal{W}) \end{aligned} \quad (3.1)$$

where

$$\begin{aligned} \mathcal{M}_{\mathcal{V}} \bigwedge \mathcal{M}_{\mathcal{W}} &= \begin{cases} \bigwedge^2 \mathcal{M}_{\mathcal{V}} & \text{if } \mathcal{V} = \mathcal{W} \\ \mathcal{M}_{\mathcal{V}} \otimes \mathcal{M}_{\mathcal{W}} & \text{if } \mathcal{V} \neq \mathcal{W} \end{cases}, \\ \mathcal{V} \wedge \mathcal{W} &= \begin{cases} \bigwedge^2 \mathcal{V} & \text{if } \mathcal{V} = \mathcal{W} \\ \mathcal{V} \otimes \mathcal{W} & \text{if } \mathcal{V} \neq \mathcal{W} \end{cases}, \\ \text{Sym}(\mathcal{M}_{\mathcal{V}} \otimes \mathcal{M}_{\mathcal{W}}) &= \begin{cases} \text{Sym}(\mathcal{M}_{\mathcal{V}}) & \text{if } \mathcal{V} = \mathcal{W} \\ \mathcal{M}_{\mathcal{V}} \otimes \mathcal{M}_{\mathcal{W}} & \text{if } \mathcal{V} \neq \mathcal{W} \end{cases}, \\ \text{Sym}(\mathcal{V} \otimes \mathcal{W}) &= \begin{cases} \text{Sym}(\mathcal{V}) & \text{if } \mathcal{V} = \mathcal{W} \\ \mathcal{V} \otimes \mathcal{W} & \text{if } \mathcal{V} \neq \mathcal{W} \end{cases}, \end{aligned}$$

and where the sum runs over all pairs $(\mathcal{V}, \mathcal{W})$ of irreducible representations.

Thus the Ext-groups of the category of co-representations of $\mathcal{M}_{\widehat{\mathfrak{M}}(E)}$ labelled by an irreducible representation $\text{Sym}^n h^1(E)(a - m)$ are given by the cohomology of the complex

$$\begin{aligned} &\mathcal{M}_{\text{Sym}^n h^1(E)(-m)} \otimes \text{Sym}^n h^1(E)(-m) \xrightarrow{\bar{d}} \\ &\sum_{\text{Sym}(\mathcal{V} \otimes \mathcal{W})} (\mathcal{M}_{\mathcal{V}} \bigwedge \mathcal{M}_{\mathcal{W}}) \otimes (\text{Sym}(\mathcal{V} \otimes \mathcal{W})) \bigoplus \sum_{\mathcal{V} \wedge \mathcal{W}} (\text{Sym}(\mathcal{M}_{\mathcal{V}} \otimes \mathcal{M}_{\mathcal{W}})) \otimes (\mathcal{V} \wedge \mathcal{W}) \\ &\longrightarrow \dots \end{aligned} \quad (3.2)$$

where \mathcal{V} and \mathcal{W} are irreducible representations, the first sum runs through all pairs $(\mathcal{V}, \mathcal{W})$ such that

$$\mathrm{Sym}^n h^1(E)(a - m) \subset \mathrm{Sym}(\mathcal{V} \otimes \mathcal{W})$$

is a summand in a direct sum decomposition of $\mathrm{Sym}(\mathcal{V} \otimes \mathcal{W})$ into irreducible representations, and the second sum runs through all pairs $(\mathcal{V}, \mathcal{W})$ such that

$$\mathrm{Sym}^n h^1(E)(a - m) \subset \mathcal{V} \wedge \mathcal{W}$$

is a summand in a direct sum decomposition of $\mathcal{V} \wedge \mathcal{W}$ into irreducible representations.⁴

It is easy to show that

$$\mathrm{Ext}_{\mathcal{M}(E)}^1(h^1(E), \mathbb{Q}) \cong E(k) \otimes \mathbb{Q}. \quad (3.3)$$

The goal of the next two sections is to prove the following proposition and remark on its consequences.

Proposition 3.1. *Let $a, n, m \in \mathbb{Z}, n, m \geq 0$. Assume that $\mathcal{A}_{\mathfrak{M}(E)}$ is cohomologically connected with respect to i . Then the appropriate component of the kernel of the map \bar{d} in (3.2) is isomorphic to*

$$(CH^{m+n-a}(E^n, 2m + n - 2a - 1) \otimes \mathbb{Q})_{\mathrm{sgn}} \boxtimes \mathrm{Sym}^n h^1(E)(a - m)$$

where sgn denotes the sign character eigenspace for the natural action of the symmetric group on E^n .

3.2 The Cycle Group Computation

In general, the decomposition of the tensor product of two projectors into projectors is complicated. However, if $\mathbb{S}_\lambda V = \varphi_1 V^{\otimes |\lambda|}$ and $\mathbb{S}_\kappa V = \varphi_2 V^{\otimes |\kappa|}$ denote two irreducible $GL(V)$ -representations (with corresponding projectors φ_1 and φ_2), then

$$\mathbb{S}_\lambda V \otimes \mathbb{S}_\kappa V = \varphi_1 V^{\otimes |\lambda|} \otimes \varphi_2 V^{\otimes |\kappa|} = (\varphi_1 \otimes \varphi_2) V^{\otimes |\lambda| + |\kappa|}$$

and there is a standard decomposition

$$(\varphi_1 \otimes \varphi_2) \cong \sum \tilde{\xi} \quad (3.4)$$

⁴I thank A. Levin for pointing out an error in the definition of complex 3.2 in a preliminary version of this paper.

(see for example [17]) where the sum runs over a certain set of projectors⁵ to irreducible representations associated to Young diagrams of size $|\lambda| + |\kappa|$. Let $\xi = \tilde{\xi} \otimes \text{Alt}$ where $\tilde{\xi} \in \mathbb{Q}[\Sigma_{|\lambda|+|\kappa|}]$ is a projector. Let $\mathcal{Z}^{a,b} = \sum_i \mathcal{Z}_{b-2a-i}^{b,a}$. We get a map of GL_2 -representations

$$(\mathcal{Z}^{a,b} \boxtimes (\rho_1 \otimes \rho_2) \cdot h_a^b) \xrightarrow{\sim} (\mathcal{Z}^{a,b} \boxtimes \sum \xi \cdot h_a^b) = \sum (\mathcal{Z}^{a,b} \cdot \xi \boxtimes \xi \cdot h_a^b)$$

Consequently we get a decomposition

$$\begin{aligned} \mathcal{A}_{\mathfrak{M}(E)}^i(a, b) &\cong \sum_{\text{Sym}^n h^1(E)(a-m)} \rho_{n,m}^t(\mathcal{Z}^{a,b}(E^b, b-2a-i)) \otimes \text{Sym}^n h^1(E)(a-m) \\ &(:= \sum_{\text{Sym}^n h^1(E)(a-m)} \mathcal{Z}^b(E^b, b-2a-i) \boxtimes \text{Sym}^n h^1(E)(a-m)) \end{aligned} \quad (3.5)$$

where we have chosen⁶ a decomposition of $h^b(E^b)$ into irreducible representations via projectors $\varrho_{n,m}$ ($n+2m=b$) and $\sum_{\text{Sym}^n h^1(E)(a-m)}$ denotes the sum over all of the irreducible representations of that decomposition.

Since $\bigwedge^n h^1(E) = 0$ for $n \geq 3$, it follows that only one term in the decomposition of $(\varrho_{0,1} \otimes \varrho_{n,m-1}) \cdot h^{n+2m}$ defines a nonzero projection. Hence we get an isomorphism of linear GL_2 -representations $(\varrho_{0,1} \otimes \varrho_{n,m-1}) \cdot h(E^2 \times E^{n+2m-2}) \cong \varrho_{n,m} \cdot h(E^{n+2m})$.

When we pass to the cycle side, however, notice that we have an *equality* $\varrho_{n,m}^t \cdot (\varrho_{0,1}^t \otimes \varrho_{n,m-1}^t)(\mathcal{Z}^{n+2m}) = \varrho_{n,m}^t(\mathcal{Z}^{n+2m})$. It follows by induction that we have equalities

$$\begin{aligned} \varrho_{n,m}^t(\mathcal{Z}^{n+2m}) &= \varrho_{n,m}^t \cdot (\varrho_{0,1}^t \otimes \varrho_{n,m-1}^t)(\mathcal{Z}^{n+2m}) = \dots \\ &= \varrho_{n,m}^t \cdot ((\varrho_{0,1}^t)^{\otimes m} \otimes \varrho_{n,0}^t)(\mathcal{Z}^{n+2m}). \end{aligned}$$

We will show that Proposition 3.1 is a consequence of the following lemma:

Lemma 3.2. $CH^{2m+n}(E^{2m+n}, 2m+n-1) \boxtimes \text{Sym}^n h^1(E)(-m) \cong (\prod_i (\Delta_i - \Psi_i)) \cdot CH^{n+m}(E^n, 2m+n-1) \cdot \varrho_{n,0} \boxtimes \text{Sym}^n h^1(E)(-m)$,
where Δ_i denotes the i -th diagonal, Ψ_j denotes the j -th antidiagonal⁷, and product is cycle-theoretic intersection.

⁵given up to Young diagrams i.e. $\mathbb{S}_\lambda V \otimes \mathbb{S}_\kappa V \cong \oplus_{\tilde{\xi}} N_{\lambda\kappa\tilde{\xi}} \mathbb{S}_{\tilde{\xi}} V$, where the numbers $N_{\lambda\kappa\tilde{\xi}}$ are given by the *Littlewood-Richardson rule*

⁶see Section 4.2 for an explicit choice of projectors $\varrho_{n,m}$

⁷If $P \in E^{2a}$ is a closed point of form $(x_1, x_2, \dots, x_{2a})$, then $\Delta_i \subset E^{2a}$ is the set of all P for which $x_i - x_{i+1} = 0$, and $\Psi_j \subset E^{2a}$ is the set of all P for which $x_j + x_{j+1} = 0$

Proof. We have the following commutative diagram :

$$\begin{array}{ccc}
& CH^{2m+n}(E^{2m+n}, 2m+n-1) & \\
& \downarrow \varrho_{0,1}^t \otimes \varrho_{n,m-1}^t & \\
\varrho_{n,m}^t \curvearrowright & CH^{2m+n}(E^2 \times E^{2m+n-2}, 2m+n-1) \cdot (\varrho_{0,1} \otimes \varrho_{n,m-1}) & \\
& \downarrow \varrho_{n,m}^t & \\
& CH^{2m+n}(E^{2m+n}, 2m+n-1) \cdot \varrho_{n,m} &
\end{array}$$

Notice that the $\varrho_{0,1}^t$ -projection simply requires that the switch map $\sigma_2(x, y) = (y, x)$ is the identity on the E^2 factor corresponding to $\phi_{0,1}^t$, and that the appropriate factor of $(\mathbb{Z}/2\mathbb{Z})^2 \subset (\mathbb{Z}/2\mathbb{Z})^n$ acting on $E \times E$ projects via the sign character. It follows that we wish to compute

$$(Id \otimes \varrho_{n,m-1}^t) \cdot (CH^{2m+n}(P \times E^{2m+n-2}, 2m+n-1)^{\iota_P=+1}) \boxtimes \text{Sym}^n h^1(E)(-m),$$

where $P := \text{Sym}^2(E)$ is a \mathbb{P}^1 -bundle over $J(E) = E$, and ι_P is the automorphism induced by the automorphism $\iota(x, y) = (-x, -y)$ on $E \times E$. Applying the projective bundle theorem yields

$$\begin{aligned}
CH^{2m+n}(P \times E^{2m+n-2}, 2m+n-1)^{\iota_P=+1} &\cong \\
CH^{2m+n}(E \times E^{2m+n-2}, 2m+n-1)^{\iota_E=+1} &\oplus \\
CH^{2m+n-1}(E \times E^{2m+n-2}, 2m+n-1)^{\iota_E=+1} \cdot c_1(\mathcal{O}_P(1)) & \quad (3.6)
\end{aligned}$$

(To save space I am omitting $\boxtimes \text{Sym}^n h^1(E)(-m)$ throughout this sequence of reductions.)

Since $Q := E/\iota \cong \mathbb{P}^1$ is a \mathbb{P}^1 -bundle over a point, we can again apply the projective bundle theorem to each factor in the above decomposition;

$$\begin{aligned}
CH^{2m+n-j}(Q \times E^{2m+n-2}, 2m+n-1) &\cong \\
CH^{2m+n-j}(E^{2m+n-2}, 2m+n-1) &\oplus CH^{2m+n-j-1}(E^{2m+n-2}, 2m+n-1) \cdot c_1(\mathcal{O}_Q(1)). \quad (3.7)
\end{aligned}$$

Let's review. The above computations show that we have the following decomposition:

$$\begin{aligned}
& \varrho_{0,1}^t(CH^{2m+n}(E^{2m+n}, 2m+n-1)) \\
& \cong \left(CH^{2m+n}(E^{2m+n-2}, 2m+n-1) \oplus CH^{2m+n-1}(E^{2m+n-2}, 2m+n-1)^{\oplus 2} \right. \\
& \quad \left. \oplus CH^{2m+n-2}(E^{2m+n-2}, 2m+n-1) \right). \quad (3.8)
\end{aligned}$$

Here we think of $\varrho_{0,1}^t$ as the projection $\varrho_{0,1}^t \otimes Id$.

Let Δ, Ψ denote the classes of the diagonal and antidiagonal, respectively. Notice that

$$4c_1(\mathcal{O}_P(1)) = c_1(\mathcal{O}_P(1)^{\otimes 4}) = c_1(\mathcal{O}_P(4))$$

pulls back to $\Delta \subset E \times E$ under the sum map $E \times E \rightarrow E$. Furthermore, $c_1(\mathcal{O}_Q(1))$ pulls back to $\Psi \subset E \times E$ under the composite of the sum map and the projection $E \rightarrow Q \rightarrow \text{pt}$. Since we are working with rational coefficients⁸, we can choose the pullback of (3.8) to $CH^{2m+n}(E^{2m+n}, 2m+n-1)$ to be

$$\begin{aligned}
& CH^0(E^2) \cdot CH^{2m+n}(E^{2m+n-2}, 2m+n-1) \\
& \oplus \Delta \cdot CH^{2m+n-1}(E^{2m+n-2}, 2m+n-1) \oplus \Psi \cdot CH^{2m+n-1}(E^{2m+n-2}, 2m+n-1) \\
& \oplus \Delta \cdot \Psi \cdot CH^{2m+n-2}(E^{2m+n-2}, 2m+n-1). \quad (3.9)
\end{aligned}$$

We now project onto the sign character eigenspace of $(\mathbb{Z}/2\mathbb{Z})^2$. First of all, notice that

$$\begin{aligned}
& \varrho_{0,1}^t(CH^0(E^2) \cdot CH^{2m+n}(E^{2m+n-2}, 2m+n-1)) = \\
& (\varrho_{0,1}^t(CH^0(E^2))) \cdot CH^{2m+n}(E^{2m+n-2}, 2m+n-1) = 0.
\end{aligned}$$

Indeed, every element of $CH^0(E^2)$ is invariant under the action of $(\mathbb{Z}/2\mathbb{Z})^2$. Thus the alternating projection is zero. Secondly, notice that

$$\Delta \cdot \Psi \cdot CH^{2m+n-2}(E^{2m+n-2}, 2m+n-1)$$

also dies under this projection. Indeed, $\Delta \cdot \Psi$ is symmetric (with respect to the action of $(\mathbb{Z}/2\mathbb{Z})^2$), not antisymmetric.

⁸in fact, all we really need in this computation is for 2 to be invertible.

Since

$$\begin{aligned}
\text{Alt}_{(\mathbb{Z}/2\mathbb{Z})^2}(\Delta) &= \text{Alt}_{(\mathbb{Z}/2\mathbb{Z})^2}((x, x)) \\
&= (x, x) - (x, -x) - (-x, x) + (x, x) \\
&= 2(\Delta - \Psi),
\end{aligned}$$

and similarly $\text{Alt}(\Psi) = 2(\Psi - \Delta)$, it follows that the sign character eigenspace of $(\mathbb{Z}/2\mathbb{Z})^2$ of the pullback of (3.8) is generated in the first two coordinates by the cycle $\Delta - \Psi$.

Finally, notice that

$$[\text{Id} + \sigma_2](\Delta - \Psi) = 2(\Delta - \Psi).$$

We have shown that

$$\begin{aligned}
&CH^{2m+n}(E^{2m+n}, 2m+n-1) \boxtimes \text{Sym}^n h^1(E)(-m) \\
&= (\Delta - \Psi) \cdot CH^{2m+n-1}(E^{2m+n-2}, 2m+n-1) \boxtimes \text{Sym}^n h^1(E)(-m). \quad (3.10)
\end{aligned}$$

The lemma now follows easily by induction on m . \square

3.3 The Case for Cohomological Connectedness, and a Theorem

In order to relate the computation of the previous section to the cohomology of our category, we need to justify why $\mathcal{A}_{\mathfrak{M}(E)}$ is quasi-isomorphic to its 1-minimal model $(\wedge(\mathcal{M}_{\mathcal{A}}[-1]))$. A sufficient condition to apply the algebraic topology machinery from the literature is that $\mathcal{A}_{\mathfrak{M}(E)}$ is connected and cohomologically connected. As remarked in Section 1, $\mathcal{A}_{\mathfrak{M}(E)}$ is connected via the Adam's grading $(2m+n)$. Cohomological connectedness, however, is somewhat more subtle. A naive approach to this question is provided by the Beilinson-Soulé conjecture. The Beilinson-Soulé conjecture implies that $CH^{n+m}(E^n, 2m+n-i) \otimes \mathbb{Q} = 0$ for $2m+n-i \geq 2m+n$, or whenever $n+i \leq 0$. It follows that $\mathcal{A}_{\mathfrak{M}(E)}$ is cohomologically connected with respect to the “total” grading $-i-n$. However, under this grading the correct Hopf algebra is difficult to see. For instance, we can no longer take H^0 of the bar complex, which we can under the more natural grading by i , since the algebra associated to a “total” grading $(i+n)$ is clearly not a $K(\pi, 1)$. We

will therefore proceed to refine our analysis of the cohomology of $\mathcal{A}_{\mathfrak{M}(E)}$ by appealing to the pure motive “labels.”

Our philosophical approach is to think of the cohomology groups

$$(CH^{n+m}(E^n, 2m+n-1) \otimes \mathbb{Q})_{sgn} \boxtimes \mathrm{Sym}^n h^1(E)(-m)$$

as extensions groups

$$\mathrm{Ext}_{\mathfrak{M}(E)}^1(\mathrm{Sym}^n h^1(E)(-m), \mathbb{Q}) \ (= \ \mathrm{Ext}_{\mathfrak{M}(E)}^1(\mathbb{Q}, \mathrm{Sym}^n h^1(E)(n+m)) \)$$

in the category of mixed elliptic motives, and as such to map under various realizations to extension groups in various target categories. In other words, we are thinking of the above cohomology groups as “cohomology with coefficients”. More precisely, let

$$H^i(Y, \mathcal{F}) := \mathrm{Ext}^i(F_Y(0), \mathcal{F})$$

denote the extension groups (thought of as cohomology) in some category of mixed sheaves with coefficient ring F (such as mixed l -adic sheaves, or even mixed motivic sheaves), where \mathcal{F} is some mixed sheaf on a scheme Y . We apply the Leray spectral sequence to the morphism $\pi : E^n \rightarrow \mathrm{Spec}(k)$. This yields a spectral sequence

$$H^r(\mathrm{Spec}(k), H^s(E^n, \mathcal{F})) \Rightarrow H^{r+s}(E^n, \mathcal{F}).$$

The idea (for $m > 0$) is to think of

$$(CH^{n+m}(E^n, 2m+n-i) \otimes \mathbb{Q})_{sgn} \boxtimes \mathrm{Sym}^n h^1(E)(-m)$$

as the E_2 term

$$H^i(\mathrm{Spec}(k), \mathrm{Sym}^n h^1(E)(-m))$$

in the spectral sequence above. Since for any cohomology theory we have $H^i(X) = 0$ for $i < 0$, we expect

$$(CH^{n+m}(E^n, 2m+n-i) \otimes \mathbb{Q})_{sgn} \boxtimes \mathrm{Sym}^n h^1(E)(-m) = 0$$

for $i < 0$, or in other words, we expect that

Conjecture 3.3. $\mathcal{A}_{\mathfrak{M}(E)}$ is cohomologically connected with respect to i .

This conjecture can be thought of as a strengthening of the Beilinson-Soulé conjecture.

We now have everything we need in order to prove Proposition 3.1. Indeed, if $\mathcal{A}_{\mathfrak{M}(E)}$ is cohomologically connected with respect to i , then by Theorem 2.30 in [8] (p. 567), $\mathcal{A}_{\mathfrak{M}(E)}$ is quasi-isomorphic to its minimal model $\wedge\mathcal{N}$. Let $\wedge\mathcal{M} := \mathcal{M}_{\mathcal{A}}[-1]$ denote the 1-minimal model of $\mathcal{A}_{\mathfrak{M}(E)}$. Now $\wedge\mathcal{M} \subset \wedge\mathcal{N}$ and $H^1(\wedge\mathcal{N}) = H^1(\wedge\mathcal{M})$ since $\wedge\mathcal{M}$ consists of all the elements of degree 1 of $\wedge\mathcal{N}$. In other words, the first cohomology group of the minimal model of $\mathcal{A}_{\mathfrak{M}(E)}$ is the same as the first cohomology group of the 1-minimal model of $\mathcal{A}_{\mathfrak{M}(E)}$. Finally notice that $\varrho_{n,0}^t$ induces the sign-character eigenspace. Hence Proposition 3.1 follows from Lemma 3.2.

In fact, we expect a stronger conjecture to be true. The minimal model machinery makes the most sense philosophically if

Conjecture 3.4. *$\mathcal{A}_{\mathfrak{M}(E)}$ is a $K(\pi, 1)$ (in the sense of Sullivan).*

In other words, we expect $\mathcal{A}_{\mathfrak{M}(E)}$ to be quasi-isomorphic to its 1-minimal model. Notice that conjecture 3.4 implies conjecture 3.3.

Furthermore, the interested reader has already noticed, no doubt, that the computations in lemma 3.2 also apply to a computation of the other cohomology groups of $\mathcal{A}_{\mathfrak{M}(E)}$. (In the case of a number field, however, we expect the higher extension groups to be zero.)

We can restate the results of this section as follows:

Theorem 3.5. *If $\mathcal{A}_{\mathfrak{M}(E)}$ is a $K(\pi, 1)$ (in the sense of Sullivan), then the cohomology of our category $\mathfrak{M}(E)$ agrees with the expected motivic cohomology groups.*

Remark 3.6. *Our construction of \mathcal{M} is well defined even if $\mathcal{A}_{\mathfrak{M}(E)}$ does not satisfy the above conjectures. Therefore, the cohomology of our category $\mathfrak{M}(E)$ makes sense independently of any conjectures.*

4 Nontrivial Elements of the Hopf Algebra χ

We will now construct families of cycle classes in each $\mathrm{Sym}^n h^1(E)(-1)$ -graded piece of the Hopf algebra $\chi = H^0(B(\mathcal{A}_{\mathfrak{M}(E)}^\bullet))$. These classes will be written out explicitly using a (non-canonical) set of choices for the projectors involved.

4.1 Some Functions

Let $0 \in E(k)$ denote the zero element under the addition law on the elliptic curve. For $1 \leq i \leq n$ let $p_i : E^n \rightarrow E$ denote projection to the i th component of E^n . Let $p_{n+1} := -\sum_{i=1}^n p_i$. Define the divisors

$$\bar{D}_i^{(n)} = p_i^*(0), \quad i = 1, 2, \dots, n+1,$$

$$\Delta_{i,j}^{(n)} = \{P \in E^n \mid p_i(P) = p_j(P)\}, \quad i, j = 1, 2, \dots, n+1, i \neq j.$$

Now define $\bar{F}_n(x, y_1, \dots, y_n) \in k(E^n)$, $n \geq 2$, by the following divisor.

$$(\bar{F}_n) = -(n) \sum_{i=1}^n \bar{D}_i^{(n)} + \sum_{1 \leq i < j \leq n} \Delta_{i,j}^{(n)} + \bar{D}_{n+1}^{(n)}$$

For example, $\bar{F}_2(x, y) \in k(E^2)$ is defined by the divisor

$$(\bar{F}_2) = \Delta_E + \Psi - 2\{E \times \{0\}\} - 2\{\{0\} \times E\}; \quad \bar{F}_2(x, y) = \bar{F}_2(y, x). \quad (4.1)$$

where Δ and Ψ denote the diagonal and the antidiagonal on E^2 respectively.

A. Levin starts with a choice of functions \bar{F}_n , $n \geq 2$ in [25], which he defines using a Vandermonde determinant of copies of the Weierstrass \mathcal{P} function, and uses associated functions to define his symbols in K-theory. It would be interesting to relate Levin's symbols to the cycle classes defined below.

4.2 Notation for Projectors

In this paper we use two related constructions of all irreducible $GL(V)$ -representations. The first standard construction is Weyl's construction of the Schur functor. See [17], for example, for an introduction to the connection between projectors in $\mathbb{C}[\Sigma_n]$, Young Tableaux, and irreducible representations of Gl_n . The second standard construction is the construction of the Spect module. A good reference for the connection between Young Tabloids and irreducible representations of Gl_n is [16].

Let $b = k + 2l$. Let T denote either a Young tableau or a Young tabloid. Let $R(T)$ and $C(T)$ denote the following subgroups of Σ_b :

$$\begin{aligned} R(T) &= \{g \in \Sigma_b \mid g \text{ preserves each row of } T\} \\ C(T) &= \{g \in \Sigma_b \mid g \text{ preserves each column of } T\} \end{aligned}$$

Define the following elements of the group algebra $\mathbb{C}\Sigma_b$:

$$c_b = \sum_{g \in R(T)} g \quad \text{and} \quad d_b = \sum_{h \in C(T)} (\text{sgn}(h))h.$$

If T is a tableau, let $\varrho_{k,l}$ denote the projector $c_b \cdot d_b \in \mathbb{C}\Sigma_b$; when T is a tabloid, define the projector $\rho_{k,l}$ to be $c_b \in \mathbb{C}\Sigma_b$.

Notation Let $\varrho_{n,m}$ correspond to the Young tableau in Figure 1, and let $\rho_{n,m}$ correspond to the column tabloid in Figure 2.

1	3	2m-1	2m+1	2m+2	2m+n
2	4	2m			

Figure 1: Young Tableau for $\varrho_{n,m}$

1	3	2m-1	2m+1	2m+2	2m+n
2	4	2m			

Figure 2: Column Tabloid for $\rho_{n,m}$

Recall that the transpose ϱ^t of a Young symmetrizer is defined by switching the roles of $R(T)$ and $C(T)$ in the construction of the element in the group algebra. Pictorially, the map $\varrho \mapsto \varrho^t$ corresponds to flipping a Young diagram about its diagonal while keeping track of the inscribed tableau.

Note that a different choice of tableau will result in different projectors thought of as elements in the group ring $\mathbb{Q}[\Sigma_n]$.

Similarly, we define the transpose of a column tabloid to be the row tabloid associated to the transpose of a representative Young tableau.

It is true that two projectors with the same Young diagram but different tableau will determine isomorphic GL_2 -representations. However, we wish to apply the projectors to arbitrary vector spaces (in our case to algebraic cycle groups), so in our context we will need to keep track of the choice of projector. Note that a different choice of projector will necessitate different choices of cycles. When we do compute with algebraic cycles, we will assume,

unless otherwise stated, that $\text{Sym}^k h^1(E)(-l)$ is given either by the projector $\varrho_{k,l}$ or $\rho_{k,l}$ (where the choice will be clear from context.)

4.3 E -motives

Choose n functions $g_1, \dots, g_n \in k(E)^*$, whose divisors are pairwise disjointly supported. Note that this necessitates using at least $2n$ distinct points of $E(k)$. Furthermore, notice that in order for the cycle below to be defined either all divisors of the g_i 's must be disjoint from $\{(0)\}$ or we need to modify \bar{F}_n slightly to avoid inadmissibility. A necessary condition for the cycles below to be nontrivial is that the functions g_i should not be even.⁹

For future reference: Let $u, v, u + v \in E(k)$ denote the nonzero closed points of order two. Choose functions $h_n \in k(E), n \in \mathbb{Z}^+, n > 1$ that have the following divisors

$$(h_n) := \begin{cases} n(u) - n(0) & \text{if } n \text{ is even} \\ (n-2)(u) + (v) + (u+v) - n(0) & \text{if } n \text{ is odd} \end{cases}$$

Define

$$F_n(z_1, \dots, z_n) := \bar{F}_n(z_1, \dots, z_n) h_n^{-1}(z_2) \cdots h_n^{-1}(z_n)$$

In other words, define divisors

$$D_i^{(n)}(u) = p_i^{(n)*}(u), \quad i = 1, 2, \dots, n,$$

Then

$$(F_n) = -(n) \sum_{i=1}^n D_i^{(n)}(u) + \sum_{1 \leq i < j \leq n} \Delta_{i,j}^{(n)} + D_{n+1}^{(n)}(0)$$

for n even and

$$\begin{aligned} (F_n) &= -(n-2) \sum_{i=1}^n D_i^{(n)}(u) - \sum_{i=1}^n D_i^{(n)}(v) - \sum_{i=1}^n D_i^{(n)}(u+v) \\ &\quad + \sum_{1 \leq i < j \leq n} \Delta_{i,j}^{(n)} + D_{n+1}^{(n)}(0) \end{aligned}$$

for n odd.

For example, $F_2(x, y) \in k(E^2)$ is defined by the divisor

$$(F_2) = -2\{E \times \{u\}\} - 2\{\{0\} \times E\} + \Delta + \Psi$$

⁹ g_i is said to be even if $g_i(-x) = g_i(x)$

where $\Delta = \{(x, x) | x \in E(k)\}$ denotes the diagonal on $E \times E$, and $\Psi = \{(x, -x) | x \in E(k)\}$ denotes the antidiagonal.

Define cycles $X_{F_{n+1+r}, g_1, \dots, g_n}^{a_1, \dots, a_r}$, Y_{g_1, \dots, g_n}^a , and ${}_jZ_{g_1, \dots, g_n}^{b_1, b_2}$ parametrically as follows:

$$\begin{aligned} X_{F_{n+1+r}, g_1, \dots, g_n}^{a_1, \dots, a_r} = & \left\{ \left(x, \left(-x - \sum_{i=1}^n y_i - \sum_{j=1}^r a_j \right), y_1, \dots, y_n, \right. \right. \\ & \left. \left. F_{n+1+r}(x, y_1, \dots, y_n, a_1, \dots, a_r), g_1(y_1), \dots, g_n(y_n) \right) \right. \\ & \left. \mid (x, y_1, \dots, y_n) \in E^{n+1}, a_j \in E(k), 1 \leq j \leq r \right\} \\ & \in \mathcal{Z}^{n+2}(E^{n+2}, n+1) \end{aligned} \quad (4.2)$$

$$\begin{aligned} Y_{g_1, \dots, g_n}^a := & \left\{ \left(\left(-\sum_{i=1}^n y_i - a \right), y_1, \dots, y_n, g_1(y_1), \dots, g_n(y_n) \right) \right. \\ & \left. \mid (y_1, \dots, y_n) \in E^n, a \in E(k), 1 \leq j \leq p \right\} \\ & \in \mathcal{Z}^{n+1}(E^{n+1}, n) \end{aligned} \quad (4.3)$$

$$\begin{aligned} {}_jZ_{g_1, \dots, g_n}^{b_1, b_2} := & \left\{ \left(x, \left(-x - \sum_{i=1, i \neq j}^n y_i - b_1 - b_2 \right), y_1, \dots, \hat{y}_j, \dots, y_n, g_1(y_1), \dots, g_j(b_2), \dots, g_n(y_n) \right) \right. \\ & \left. \mid (x, y_1, \dots, \hat{y}_j, \dots, y_n) \in E^n, b_1, b_2 \in E(k), 1 \leq j \leq n \right\} \\ & \in \mathcal{Z}^{n+1}(E^{n+1}, n) \end{aligned} \quad (4.4)$$

where \hat{g}_j or \hat{y}_i denotes omission.

Let $\langle A \rangle$ denote the \mathbb{Q} -vector space spanned by A.

Definition 4.1. *We have*

$$\begin{aligned}
\eta_{\mathrm{Sym}^n h^1(E)(-1)}^{a_1, \dots, a_r}(g_1, \dots, g_n) &:= \langle \rho_{n,1}^t(X_{F_{n+1+r}, g_1, \dots, g_n}^{a_1, \dots, a_r}) \rangle \boxtimes \rho_{n,1} \cdot (h^1(E))^{\otimes n+2} \\
&\subset \mathcal{Z}^{n+2}(E^{n+2}, n+1) \boxtimes \mathrm{Sym}^n h^1(E)(-1) \\
(\text{In particular, } \eta_{h^1(E)}(p) &:= \langle ((p) - (-p)) \rangle \otimes h^1(E) \subset \mathcal{Z}^1(E) \boxtimes h^1(E)) \\
\mu_{\mathrm{Sym}^{n+1} h^1(E)}^a(g_1, \dots, g_n) &:= \langle Y_{g_1, \dots, g_n}^a \rangle \boxtimes \mathrm{Sym}^{n+1} h^1(E) \\
&\subset \mathcal{Z}^{n+1}(E^{n+1}, n) \boxtimes \mathrm{Sym}^{n+1} h^1(E) \\
j\nu_{\mathrm{Sym}^{n-1} h^1(E)(-1)}^{b_1, b_2}(g_1, \dots, g_n) &:= \langle \rho_{n-1,1}^t(jZ_{g_1, \dots, g_n}^{b_1, b_2}) \rangle \boxtimes \rho_{n-1,1} \cdot h^1(E)^{\otimes n+1} \\
&\subset \mathcal{Z}^{n+1}(E^{n+1}, n) \boxtimes \mathrm{Sym}^{n-1} h^1(E)(-1)
\end{aligned}$$

Remark 4.2. *Since*

$$\bar{F}_n|_{p_i^{(n)*}(0)} = \bar{F}_{n-1} \text{ for } 1 \leq i \leq n,$$

the following cycle is well defined:

$$\eta_{\mathrm{Sym}^n h^1(E)(-1)} := \eta_{\mathrm{Sym}^n h^1(E)(-1)}^{0, \dots, 0}(g_1, \dots, g_n).$$

When one unwinds the various definitions one finds that

$$\partial \eta_{\mathbb{Q}(-1)}^{a_1, \dots, a_r} = \sum_r \delta \left[\eta_{h^1(E)}(a_i) \otimes \eta_{h^1(E)}(-a_i - \sum_{j=1}^r a_j) \right]. \quad (4.5)$$

and

$$\begin{aligned}
&\partial \eta_{\mathrm{Sym}^n h^1(E)(-1)}^{a_1, \dots, a_r}(g_1, \dots, g_n) \\
&= \delta \left[\sum_i \sum_{p \in \mathrm{div}(g_i)} \eta_{\mathrm{Sym}^{n-1} h^1(E)(-1)}^{a_1, \dots, a_r, p}(g_1, \dots, \hat{g}_i, \dots, g_n) \otimes (\eta_{h^1(E)}(p)) \right. \\
&\quad + \sum_{l=1}^r \eta_{h^1(E)}(a_l) \otimes \mu_{\mathrm{Sym}^n h^1(E)}^{a_1 + \dots + 2a_l + \dots + a_r}(g_1, \dots, g_n) \\
&\quad \left. + \sum_l \sum_j j\nu_{\mathrm{Sym}^{n-1} h^1(E)(-1)}^{a_1 + \dots + a_r, a_l}(g_1, \dots, g_n) \otimes \eta_{h^1(E)}(a_l) \right]. \quad (4.6)
\end{aligned}$$

Also note that

$$\begin{aligned} & \partial(\mu_{\mathrm{Sym}^n h^1(E)}^a(g_1, \dots, g_n)) \\ &= \sum_i \sum_{p \in \mathrm{div}(g_i)} \delta[\mu_{\mathrm{Sym}^{n-1} h^1(E)}^{a+p}(g_1, \dots, \hat{g}_i, \dots, g_n) \otimes (\eta_{h^1(E)}(p))] \quad (4.7) \end{aligned}$$

Furthermore,

$$\partial({}_j\nu_{\mathrm{Sym}^{n-1} h^1(E)(-1)}^{a,b}(g_1, \dots, g_n)) = 0,$$

since the boundary of this cycle dies under the alternating condition on the $\mathbb{P}^1 \setminus \{1\}$ coordinates. Indeed, let $\sigma_{i,j}$ be the transposition that switches the i th and j th $\mathbb{P}^1 \setminus \{1\}$ -coordinate. Then

$$\partial({}_j Z_{g_1, \dots, g_n}^{a,b} - \sigma_{i,j}({}_j Z_{g_1, \dots, g_n}^{a,b})) = 0.$$

Thus we have shown

Proposition 4.3. *∂C of an element C in the subring R generated by the $\eta_{\mathrm{Sym}^n h^1(E)(-1)}^{a_1, \dots, a_r}$'s, the $\mu_{\mathrm{Sym}^{n+1} h^1(E)}^a$'s, and the ${}_j\nu_{\mathrm{Sym}^{n-1} h^1(E)(-1)}^{b_1, b_2}$'s (where $n \geq 0$ and $a, b_1, b_2, a_1, \dots, a_r$ are closed points of E) is the image under the product map of a linear combination of cycles again in R .*

However, the situation is much simpler than it first appears.

Proposition 4.4. *The $\mu_{h^1(E)^{n+1}(-n-1)}^a(g_1, \dots, g_n)$ and ${}_j\nu_{\mathrm{Sym}^{n-1} h^1(E)}^{b_1, b_2}(g_1, \dots, g_n)$ terms appearing in the “successive boundary” terms of $\eta_{\mathrm{Sym}^n h^1(E)(-1)}$ are trivial $\in H^0(B(\mathcal{A}_{\mathfrak{M}(E)}))$.*

Sketch of proof:

We have already remarked above that the boundary of ${}_j\nu_{\mathrm{Sym}^{n-1} h^1(E)}^{b_1, b_2}(g_1, \dots, g_n)$ is zero. The terms of this cycle appearing in the successive boundaries of $\eta_{\mathrm{Sym}^n h^1(E)(-1)}$ are killed by cycles of the form

$$\left\{ \left(x, \left(-x - \sum_{i=1}^n y_i - b_1 \right), y_1, \dots, \hat{y}_j, \dots, y_n, g_1(y_1), \dots, g_j(y_j), \dots, g_n(y_n), g_j(b_2) \right) \right\}$$

where b_2 is in the divisor of g_j .

For example, $\sum_{a \in (g_1)} (y, -y - a, g_2(b))$ is killed by $(y, y - z, g_1(z), g_2(b))$.

The case of $\mu_{h^1(E)^{n+1}(-n-1)}^a(g_1, \dots, g_n)$ is a bit more involved. One shows that the final term of its successive boundaries is zero, (hence the class defined by such cycles does actually go to zero) and then one can kill it with terms of the form

$$\left\{ \left(-\sum_{i=1}^n y_i - z, y_1, \dots, y_n, g_1(y_1), \dots, g_n(y_n), g_i(z) \right) \right. \\ \left. \mid (y_1, \dots, y_n) \in E^n, 1 \leq i \leq n \right\}$$

For example, the cycle $(x, -x - y, g(x), g(y))$ and its successive boundary kills the cycle $\sum_{a \in (g)} (x, -x - a, g(x))$ and its successive boundary.

End of sketch of proof.

Hence we can use $\eta_{\text{Sym}^n h^1(E)(-1)}$ and its “successive boundaries” to define a cohomology class $\in H^0(B(\mathcal{A}_{\mathfrak{M}(E)}))$. Explicitly,

$$\mathcal{E}(g_1, \dots, g_n) := \left\{ \eta_{\text{Sym}^n h^1(E)(-1)}(g_1, \dots, g_n), \right. \\ \sum_i \sum_{p \in \text{div}(g_i)} [\eta_{\text{Sym}^{n-1} h^1(E)(-1)}^p(g_1, \dots, \hat{g}_i, \dots, g_n) \otimes (\eta_{h^1(E)}(p))], \\ \dots, \\ \left. \sum_{j=1}^{n+2} \bigotimes \eta_{h^1(E)}(b_j) \right\} \quad (4.8)$$

defines¹⁰ a family of cohomology classes $\in H^0(B(\mathcal{A}_{\mathfrak{M}(E)}))$.

Define

$$\mathcal{E}^{a_1, \dots, a_r}(g_1, \dots, g_n) := \left\{ \eta_{\text{Sym}^n h^1(E)(-1)}^{a_1, \dots, a_r}(g_1, \dots, g_n), \right. \\ \left[\sum_i \sum_{p \in \text{div}(g_i)} [\eta_{\text{Sym}^{n-1} h^1(E)(-1)}^{a_1, \dots, a_r, p}(g_1, \dots, \hat{g}_i, \dots, g_n) \otimes (\eta_{h^1(E)}(p))] \right] \\ \dots, \\ \left. \sum_{j=1}^{n+2} \bigotimes \eta_{h^1(E)}(c_j) \right\}, \quad (4.9)$$

¹⁰Note: the b_j 's and the limits for the sum in the final term are completely determined by the divisors of the g_i 's.

Let $[p]$ denote the class in $H^0(B(\mathcal{A}_{\mathfrak{M}(E)}))$ defined by $\eta_{h^1(E)}(p)$. It follows from our explicit calculations that

$$\begin{aligned} \psi(\mathcal{E}(g_1, \dots, g_n)) = & \left\{ \mathcal{E}(g_1, \dots, g_n) \otimes 1 \right\} + \\ & \left\{ \sum_i \sum_{p \in \text{div}(g_i)} \mathcal{E}^p(g_1, \dots, \hat{g}_i, \dots, g_n) \otimes [p] \right\} + \\ & \left\{ \sum_i \sum_{p \in \text{div}(g_i)} \sum_j \sum_{q \in \text{div}(g_j)} \mathcal{E}^{p,q}(g_1, \dots, \hat{g}_i, \dots, \hat{g}_j, \dots, g_n) \otimes ([p] \otimes [q]) \right\} + \\ & \dots, \\ & + \left\{ 1 \otimes \mathcal{E}(g_1, \dots, g_n) \right\} \quad (4.10) \end{aligned}$$

where $\hat{}$ denotes omission and where ψ denotes the co-multiplication map

$$\psi : H^0(B(\mathcal{A})) \rightarrow H^0(B(\mathcal{A})) \otimes H^0(B(\mathcal{A})).$$

Hence $\eta_{\text{Sym}^n h^1(E)(-1)}, \{ \eta_{\text{Sym}^{n-1} h^1(E)(-1)}^p(g_1, \dots, \hat{g}_i, \dots, g_n) \mid 1 \leq i \leq n \text{ and } p \in \text{div}(g_i) \}, \dots, \{ \eta_{h^1(E)}(p) \mid p \in \text{div}(g_i) \text{ for some } i \}$ and 1 span a $H^0(B(\mathcal{A}_{\mathfrak{M}(E)}))$ -comodule, and hence define a motive.

Finally, notice that, for any vector space V , $\varrho_{n,1}^t \cdot \rho_{n,1}^t(V^{\otimes n+2}) = \varrho_{n,1}^t(V^{\otimes n+2})$. In particular, if $\rho_{n,1}^t(V^{\otimes n+2}) \neq 0$, then $\varrho_{n,1}^t \cdot \rho_{n,1}^t(V^{\otimes n+2}) \neq 0$. Hence if $\mathcal{E}^{a_1, \dots, a_r}(g_1, \dots, g_n)$ defines a nontrivial class in χ , then

$$\begin{aligned} & \text{Hom}_{GL(h^1(E))}(\text{Sym}^n h^1(E)(-1), \mathcal{E}^{a_1, \dots, a_r}(g_1, \dots, g_n)) \\ & = \text{Hom}_{GL(h^1(E))}(\varrho_{n,1}^t \cdot h^{\otimes n+2}, \mathcal{E}^{a_1, \dots, a_r}(g_1, \dots, g_n)) \quad (4.11) \end{aligned}$$

defines a nontrivial class in χ . Thus a suitable linear combination of classes of this form will determine elements in

$$\begin{aligned} & \text{Ker} \left(\mathcal{M}_{\text{Sym}^n h^1(E)(-1)} \otimes \text{Sym}^n h^1(E)(-1) \xrightarrow{\bar{d}} \right. \\ & \left. \mathcal{M}_{\text{Sym}^{n-1} h^1(E)(-1)} \otimes \text{Sym}^{n-1} h^1(E)(-1) \otimes \mathcal{M}_{h^1(E)} \otimes h^1(E) \longrightarrow \dots \right) \quad (4.12) \end{aligned}$$

Given a suitable description of a realization functor to the category of mixed Hodge structures, we expect such elements to determine nonzero multiples of $L(\text{Sym}^n E, n+1)$.

Notice that the final term $\sum \bigotimes_{j=1}^{n+2} \eta_{h^1(E)}(c_j)$ of $\mathcal{E}(g_1, \dots, g_n)$ is generically not a coboundary, since, for a point $P \in E(k)$, $(P) - (-P)$ is not the divisor of a function. It follows therefore that the element $\mathcal{E}(g_1, \dots, g_n)$ itself is nontrivial. Thus we have shown

Theorem 4.5. $\mathcal{M}_{\mathcal{U}} \neq (0)$ for pure motives $\mathcal{U} = \text{Sym}^n h^1(E)(-1)$, n any natural number.

In particular, our category is nontrivial!

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